Variational measures and descriptive characterization of generalized integrals

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Motivation

There are many areas in harmonic analysis which require non-absolutely convergent integration processes more powerful than the Lebesgue integration.
In particular Denjoy-Perron and Kurzweil-Henstock type integrals, defined with respect to various derivation bases, are used to solve the problem of recovering, by generalized Fourier formulae, the coefficients of orthogonal series.

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There are many areas in harmonic analysis which require non-absolutely convergent integration processes more powerful than the Lebesgue integration.
In particular Denjoy-Perron and Kurzweil-Henstock type integrals, defined with respect to various derivation bases, are used to solve the problem of recovering, by generalized Fourier formulae, the coefficients of orthogonal series.

In turn the problem of recovering the coefficients is usually reduced to the problem of recovering a primitive from its derivative where the differentiation is understood in the respective sense.

Descriptive characterization in a classical case

The known descriptive characterization of the indefinite Lebesgue integral in terms of absolutely continuous functions is equivalent to the following statement:

Characterization of L-integral in terms of measure

A function f is L-integrable on $[a, b]$ if and only if there exists a function F of bounded variation on $[a, b]$ which generates an absolutely continuous Lebesgue-Stieltjes measure and $F'(x) = f(x)$ a.e.; the function $F(x) - F(a)$ being the indefinite L-integral of f.

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In case of non-absolute generalizations of the Lebesgue integrals (of Denjoy-Perron- or Henstock-type) indefinite integrals fail to be of bounded variation and so cannot generate a finite Stieltjes measure.

In this case a descriptive characterization can be obtained in terms of some generalized σ-finite outer measure (so-called variational measure) generated by the indefinite integral.

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B. Bongiorno, L. di Piazza, V. Skvortsov, *A new full descriptive characterization of Denjoy-Perron integral*, Real Analysis Exchange vol. 20 (2) 1995 - 1996, pp. 656 - 663.

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$$\beta_\delta := \{(I, \mathbf{x}) : I \in \mathcal{I}, \quad \mathbf{x} \in I \subset U(\mathbf{x}, \delta(\mathbf{x}))\}$$

where δ is the so-called gauge, i.e., a positive function defined on K, and $U(\mathbf{x}, r)$ denotes the neighborhood of $\mathbf{x}$ of radius r.

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If $(I, \mathbf{x}) \in \beta_\delta$, we say that $\mathbf{x}$ is the tag of I and that the pair $(I, \mathbf{x})$ is δ-fine.

β_δ-partition

A β_δ-partition is a finite collection π of elements of β_δ, where the dist inct elements $(I', \mathbf{x}')$ and $(I'', \mathbf{x}'')$ in π have I' and I'' non-overlapping, i.e., they have no inner points in common. Let $E \in \mathcal{I}$. Then $\pi \subset \beta_\delta(E)$ is called β_δ-partition (or δ-fine β-partition) in E. If $\bigcup_{(I,\mathbf{x})\in\pi} I = E$ then π is called δ-fine β-partition of E.

Variational measure of E generated by F

Definition of variational measure

For $F : [a, b] \to \mathbf{R}$, a set $E \in [a, b]$ and a fixed gauge δ on E, we define δ-variation of F on E by

$$V_\delta(E) = Var(E, F, \delta) := \sup\{\sum_{i=1}^{k} |F(d_i) - F(c_i)|\}$$

where sup is taken over all δ-fine partition $\{[c_i, d_i], x\}_i$ in $[a, b]$ tagged in E.

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It can be checked that V_F is a metric outer measure.

Kurzweil-Henstock integral

Kurzweil-Henstock integral on the interval K.

Definition ($H_{\mathcal{B}}$-integral)

A point-function f on $K \in \mathcal{I}$ is said to be $H_{\mathcal{B}}$-integrable on K, with $H_{\mathcal{B}}$-integral A, if for every $\varepsilon > 0$, there exists a gauge δ such that for any β_δ-partition π of K we have:

$$\left| \sum_{(I,\mathbf{x})\in\pi} f(\mathbf{x})|I| - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_K f$.

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In the case of usual interval bases (i.e., when $\mathcal{I}$ is the collection of *all* closed subintervals of an interval K) the class of indefinite Henstock-Kurzweil integrals (which is equivalent in this case to Denjoy-Perron integral) coincides with the class of functions generating an absolutely continuous variational measures, see Lukashenko T.P., Skvortsov V.A., Solodov A.P., Generalized integrals, Liberkom, Moscow, 2011.

Kurzweil-Henstock integral

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Descriptive characterization of $H_{\mathcal{B}}$-integral

A function f is $H_{\mathcal{B}}$-integrable on $[a, b]$ if and only if there exists a function F on $[a, b]$ which generates an absolutely continuous variational measure, the function $F(x) - F(a)$ being the indefinite $H_{\mathcal{B}}$-integral of f with $f(x) = F'_{\mathcal{B}}(x)$ a.e.

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Definition ($\mathcal{B}$-derivative)

Given a $\mathcal{B}$-interval function F, the $\mathcal{B}$-derivatives of F at a point $\mathbf{x}$, with respect to the basis $\mathcal{B}$ is defined as

$$D_{\mathcal{B}}F(\mathbf{x}) := \lim_{\delta(\mathbf{x})\to 0}\{\frac{F(I)}{|I|} : \quad (I,\mathbf{x})\text{are } \delta\text{-fine intervals tagged in } \mathbf{x}\}.$$

F is $\mathcal{B}$-differentiable at $\mathbf{x}$ if the $\mathcal{B}$-derivative at this point exists and is finite.

L^r-variational measure

A little bit more general case we obtain if we replace usual increments in the definitions of derivatives, integrals and variations with the increments in the space $L^r[a, b]$, $r \geq 1$.

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For $F \in L^r[a,b]$ and a tagged interval (I,x), let

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For a set $E \in [a,b]$ and a fixed gauge δ on E, we define δ-variation of F on E by

$$\mathrm{Var}(E,F,\delta,r) = \sup \sum_{i=1}^{q} \Delta_r F(I_i, x_i)$$

where the sup is taken over all δ-fine partition $\{(I_i, x_i)\}$ in $[a,b]$ tagged in E.

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$$\mathrm{V}_F(E) = \inf_{\delta} \mathrm{Var}(E,F,\delta,r)$$

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L^r-Henstock-Kurzweil integral

In 2004 Musial and Sagher introduced the L^r-Henstock-Kurzweil integral (see P. Musial, Y. Sagher, The L^r Henstock-Kurzweil integral, Studia Math, vol. 160 (1) 2004)

Definition of L^r-Henstock-Kurzweil integral

A function $f : [a, b] \to \mathbb{R}$ is L^r-Henstock-Kurzweil integrable (HK_r-integrable) on $[a, b]$ if there exists a function $F \in L^r[a, b]$ so that for any $\varepsilon > 0$ there exists a gauge δ so that for any finite collection of nonoverlapping δ-fine tagged intervals $\mathcal{Q} = \{(x_i, [c_i, d_i]), 1 \leq i \leq q\}$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

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The function F in this Definition is unique up to an additive constant, so we can consider the indefinite HK_r-integral.

L^r-Henstock-Kurzweil integral

Definition of L^r-derivative ()

We say F is L^r-differentiable at x, if there exists a real number α such that

$$\left(\frac{1}{h}\int_{-h}^{h}\left|F\left(x+t\right)-F\left(x\right)-\alpha t\right|^{r}dt\right)^{\frac{1}{r}}=o\left(h\right).$$

In this case we say that α is the L^r-derivative at x and denote $F_r'(x)=\alpha$.

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This notion was introduced by Calderon and Zygmund in order to establish pointwise estimates for solutions of elliptic partial differential equation.

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In this case we say that α is the L^r-derivative at x and denote $F'_r(x)=\alpha$.

This notion was introduced by Calderon and Zygmund in order to establish pointwise estimates for solutions of elliptic partial differential equation.

It can be proved that the indefinite HK_r-integral F is L^r-differentiable a.e. and $F'_r(x)=f(x)$ a.e.

Descriptive characterization of L^r-Henstock-Kurzweil integral

A variational measure V_F is said to be absolutely continuous on a set $E \subset [a, b]$ if $\mathrm{V}_F(N) = 0$ for any $N \subset E$ such that $\mu N = 0$.

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Descriptive characterization of L^r-Henstock-Kurzweil integral (it is a joint result with P. Musial (USA) and F. Tulone (Italy)) to be published in Math. Notes

A function f is HK_r-integrable on $[a,b]$ if and only if there exists a function F on $[a,b]$ which generates an absolutely continuous L^r-variational measure and which is L^r-differentiable almost everywhere with $F'_r = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.

σ-finiteness of variational measure

Theorem

If the L^r-variational measure generated by a function $F\colon [a,b] \to \mathbb{R}$ is absolutely continuous on a closed set $E \subset [a,b]$, then V_F is σ-finite on E.

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Definition of ACG_r class

Let $E \subset [a,b]$. We say that $F \in AC_r(E)$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ defined on E so that for any finite collection of nonoverlapping δ-fine tagged intervals $\{(x_i, I_i), 1 \le i \le q\}$ with $x_i \in E$, and such that $\sum_{i=1}^{q} |I_i| < \eta$ we have

$$\sum_{i=1}^{q} \Delta_r F(I_i, x_i) < \varepsilon. \tag{1}$$

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$$\sum_{i=1}^{q} \Delta_r F(I_i, x_i) < \varepsilon. \qquad (1)$$

We say that $F \in ACG_r(E)$ if E can be written as $E = \cup_{n=1}^{\infty} E_n$ where $F \in AC_r(E_n)$ for all n.

σ-finiteness of variational measure and $ACG_r([a,b])$ class

The next two theorems show that the class $ACG_r([a,b])$ coincides with the class of functions which generate absolutely continuous L^r-variational measures.

Theorem

Suppose that the L^r-variational measure V_F generated by a function $F\colon [a,b] \to \mathbb{R}$ is finite on a set $E \subset [a,b]$. Then V_F is absolutely continuous on E if and only if $F \in AC_r(E)$.

Remark

Note that the proof of sufficiency in the above theorem does not require finiteness of V_F on E. Hence the condition $F \in AC_r(E)$ always implies absolute continuity of L^r-variational measure V_F on E.

σ-finiteness of variational measure and $ACG_r([a,b])$ class

Theorem

A function $F\colon [a,b] \to \mathbb{R}$ generates an absolutely continuous L^r-variational measure on $[a,b]$ if and only if $F \in ACG_r([a,b])$.

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Theorem

A function $F\colon [a, b] \to \mathbb{R}$ generates an absolutely continuous L^r-variational measure on $[a, b]$ if and only if $F \in ACG_r([a, b])$.

Theorem

If a function $F\colon [a, b] \to \mathbb{R}$ is L_r-differentiable almost everywhere on $[a, b]$ then it generates an absolutely continuous variational measure on $[a, b]$ if and only if it is the indefinite HK_r-integral of its L^r-derivative F'_r.

Problem

Remark

It is most likely that the a priori assumption on L^r-differentiability of F can be dropped in the above theorems. So up to now we leave open the following question: is the class of functions generating absolutely continuous L^r-variational measure (or, equivalently, the class $ACG_r[a,b]$) coincides with the class of HK_r-integral functions? In other words: is each function in those classes L^r-differentiable almost everywhere? Note that in the case of the classical Kurzweil-Henstock integral the answer to these questions is positive

Other examples of bases and variational measures w.r. to them

In classical harmonic analysis various kind of symmetric bases are useful. For example *approximate symmetric basis* in definition of which basis set are constituted by pairs $([x-h, x+h], x)$ and points $x-h$ and $x+h$ belong to a set E_x with x being a point of density for E_x.

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Approximate symmetric Henstock-type integral, corresponding to this basis, recovers, by generalized Fourier formulas, coefficients of any everywhere convergent trigonometric series.

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This integral also can be described in terms of correspondent symmetric variational measure

Other examples of bases and variational measures w.r. to them

In Dyadic Analysis the dyadic basis on $X = [0, 1]$ and on $[0, 1]^m$ is used
In the case of $[0, 1]$ the family $\mathcal{I}$ of $\mathcal{B}$-intervals is constituted by the closures of dyadic intervals

$$J_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \quad 0 \leq j \leq 2^n - 1, \quad n = 0, 1, 2, \ldots .$$

If $X = [0, 1]^m$, $\mathcal{B}$-intervals are defined as the closures of m-dimensional dyadic intervals

$$J_{\mathbf{j}}^{(\mathbf{n})} := J_{j_1}^{(n_1)} \times \ldots \times J_{j_m}^{(n_m)}$$

where $\mathbf{j} = (j_1, \ldots, j_m)$ and $\mathbf{n} = (n_1, \ldots, n_m)$.

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Dyadic Henstock integral, corresponding to this basis, is used to recover, by generalized Fourier formulas, coefficients of convergent Walsh series.

THANK YOU FOR YOUR ATTENTION!!